Some Minimal Shape Decompositions Are Nice

Kevin R. Vixie

Washington State University

In a Nutshell

In some sense, the world is composed of *shapes* and *words*, of continuous things and discrete things. The recognition and study of continuous objects in the form of shapes occupies a significant part of the effort of unraveling many geometric questions. Shapes can be represented with great generality by objects called *currents*. While the enormous variety and representational power of currents is useful for representing a huge variety of phenomena, it also leads to the problem that knowing something is a respectable current tells you little about how nice or *regular* it is. In these brief notes I give an intuitive explanation of a result that says that an important class of minimal shape decompositions will be nice if the input shape (current) is nice. *These notes are an exposition of the paper by Ibrahim, Krishnamoorthy and Vixie [3] which can be found on the arXiv: http://arxiv.org/abs/1411.0882 and any reference to these notes, should include a reference to that paper as well.*

Shapes and Currents

We begin by representing shapes as k-*currents* which, for this section, should be thought of as oriented k-dimensional subsets of $\mathbb{R}^n$ ($0 \leq k \leq n$) that are locally flat almost everywhere[1].

It turns out that currents[2] are a deep generalization of sets like 2-dimensional surfaces in $\mathbb{R}^3$ or 1-dimensional curves in $\mathbb{R}^2$ or $\mathbb{R}^3$. This is both very helpful, because we can represent a large class of things very naturally, and challenging, because knowing something is a respectable current no longer implies it is in some small class of relatively tame objects.

As a result of this diversity, we are naturally led to classify currents into different subspaces – e.g. integral, normal, or flat currents – and when we find that some problem has a current as a solution, we then occupy ourselves with finding out which of these classes that current falls into. This in turn tells us how tame (how *regular*) it is. The nicest class is the class of integral currents. Proving that a current is in a smaller, more regular set of currents can sometimes be very challenging.

---

[1]By *locally flat almost everywhere*, we mean that when we zoom in to almost any point in the current, it looks (measure theoretically) like a k-dimensional plane. Such sets are called *rectifiable sets*.

[2]To be more precise, k-currents are defined to be elements of the dual space to the space of smooth k-forms in $\mathbb{R}^n$. But for the purposes of these notes, thinking of them as unions of pieces of oriented k-manifolds will get you a long way towards understanding currents.

Shapes and Currents: Examples

Example 1 (Orientation). *Suppose you have a simple closed curve in $\mathbb{R}^2$ – it does not cross itself and is really just a circle that has (possibly) been contorted or stretched around. Then an orientation is a choice of direction along the curve: you can choose a unit tangent vector everywhere such that the directions are all consistent. Notice that there will be two possible choices, clockwise and counterclockwise. For a closed 2-dimensional surface in $\mathbb{R}^3$ – say a sphere is $\mathbb{R}^3$ – we can orient the surface by choosing little tangent planes with normal vectors pointing either inside or outside of the surface. Even if the surface is not closed, we can attempt to assign a little tangent patch and normal vector to every point in a way that is continuous. As long as all these choices are consistent (the normal vectors change continuously) then we say the surface is oriented by the little 2-planes which we call 2-vectors.*
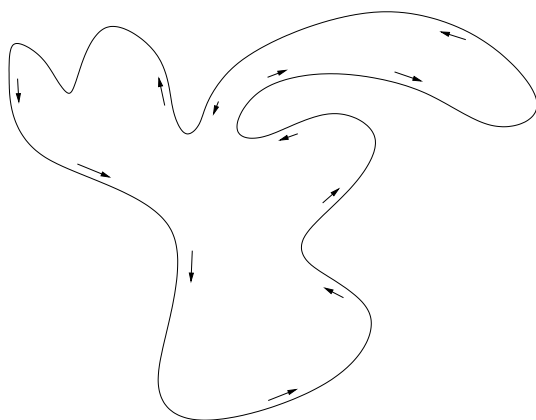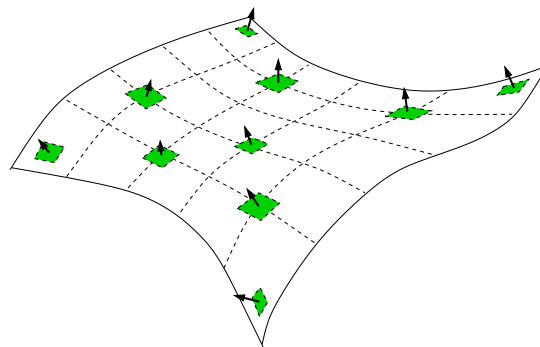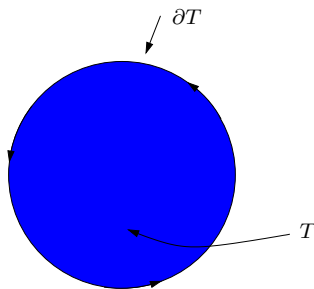

(a) Oriented Curve   (b) Oriented Surface

Example 2 (Boundaries). *We will denote the boundary of T by ∂T. If T is the 2-current in $\mathbb{R}^2$ corresponding to the unit disk with counterclockwise (right-handed) orientation, ∂T is the unit circle oriented in a counterclockwise direction.*

Example 3 (Mass of a current). *$M(T)$ denotes the* mass *of the k-current, which for the purposes of these notes should be thought of as either the k-dimensional volume of the current or as the multiplicity weighted k-dimensional volume of the current. For example, If T is the 2-current in $\mathbb{R}^2$ corresponding to the unit disk then $M(T) = \pi$ and the mass $M(\partial T) = 2\pi$. In the important case in which we have T_Ω, the current corresponding to an oriented surface Ω, we get that $M(T_\Omega) = \int_\Omega 1 dx$ as is illustrated here.*

Counterclockwise orientation

Ω

T_Ω

$M(T_\Omega) = \int_\Omega 1 dx$
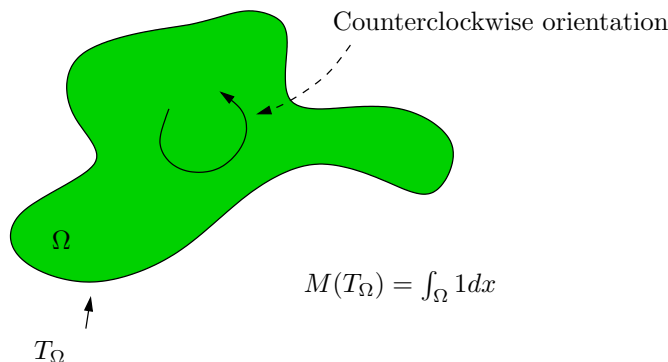
Example 4 (Adding and Subtracting Currents). *A key feature of currents is the fact that* it makes sense to add and subtract currents: *Suppose that T_1 is the unit disk T_1 oriented counterclockwise and T_2 is the intersection of the counterclockwise oriented unit disk intersected with the upper half plane. Then $T_3 = T_1 + T_2$ will be the 2-dimensional current corresponding to the unit disk with multiplicity 2 in the upper half disk, and multiplicity 1 in the lower half disk. It's boundary will be the union of the circle of unit radius whose upper half has multiplicity 2 and lower half has multiplicity 1 and the diameter segment through the origin with multiplicity 1.*
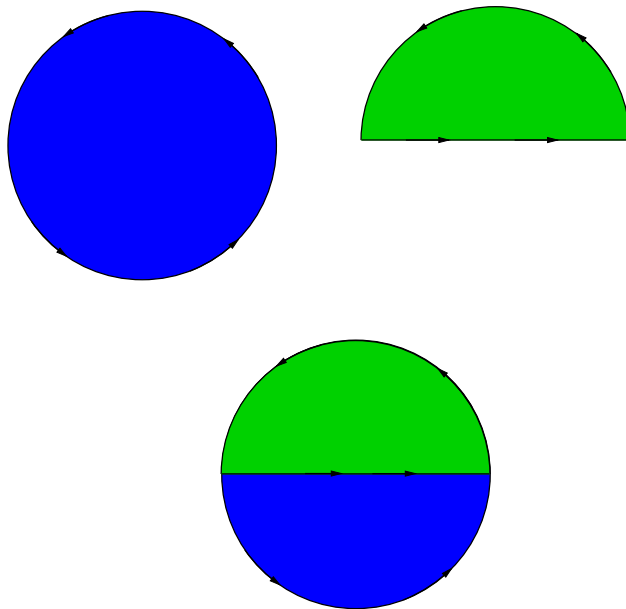
Subtracting T_2 from T_1, we simply get the lower half unit disk or lower semi-disk, again oriented counterclockwise.

Example 5 (Mass, again). *If T_1 is the counterclockwise oriented unit disk at the origin, and T_2 is the counterclockwise oriented unit disk centered at $(0, 3)$, then $T = 2T_1 + 3T_2$, is the sum of the oriented unit disk centered at the origin with multiplicity 2 and oriented the unit disk centered at $(0, 3)$ with multiplicity 3. The mass of T is 5π and $M(\partial T)$ is 10π. In general, if the multiplicity of the current is $N(x)$ then $M(T_\Omega) = \int_\Omega N(x) dx$.*

3
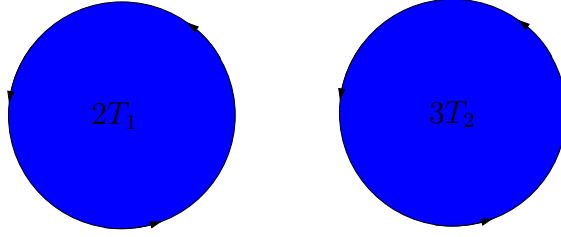
$$T = 2 * T_1 + 3 * T_2$$

$$
\begin{aligned}
M(T) &= M(2T_1 + 3T_2) \\
&= 2M(T_1) + 3M(T_2) \\
&= 5\pi
\end{aligned}
$$

## Distances Between Shapes: The Multiscale Flat Norm

A first requirement for studying collections of objects is a distance between those objects. When those objects are currents there is a very natural distance called the *flat norm*, $\mathbb{F}_\lambda(T)$. This distance begins by decomposing the input k-current T into two pieces $T = (T-\partial S)+\partial S$, where S is a k+1-current. Now we measure the cost of the decomposition as $M(T - \partial S) + \lambda M(S)$. Finally, we minimize this cost over all possible decompositions (i.e. over all possible k+1 currents S):

$$\mathbb{F}_\lambda(T) = \min_S \left(M(T - \partial S) + \lambda M(S) \right).$$

Example 6 (The multiscale flatnorm)**.** *In this example, a 1-current T is decomposed into two pieces, $T - \partial S$ and ∂S, in a way that minimizes*

$$M(T - \partial S) + \lambda M(S).$$

It turns out that in this case it can be shown that the optimal decomposition fills in the corners of T with S such that the resulting $T - \partial S$ has curvature bounded by $\frac{1}{\lambda}$; that is, we short-cut the corners with arcs of circles of radius $\frac{1}{\lambda}$.
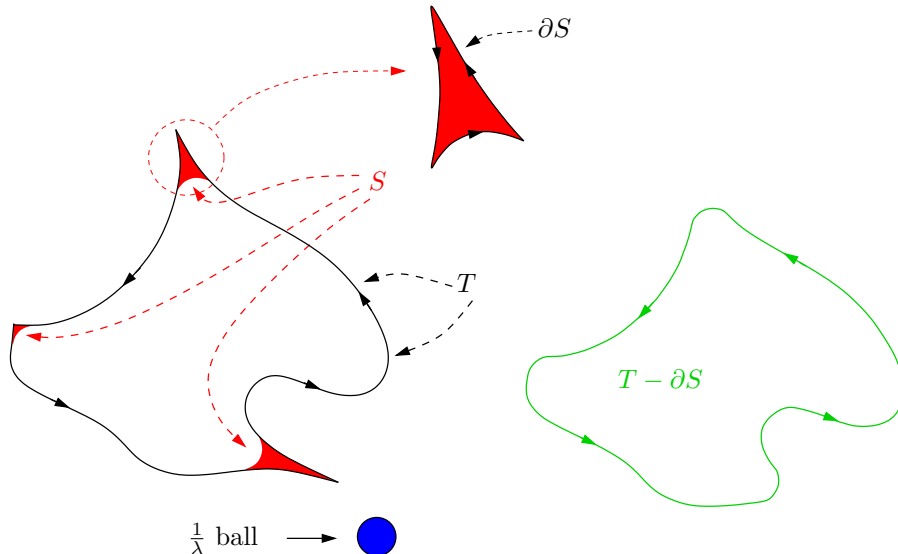
While it might seem difficult to compute the flat norm, the realization that the flat norm is connected to the L^1TV functional from image analysis[5] permits efficient calculation for the case when T is an (n-1)-dimensional boundary of an n-dimensional set in $\mathbb{R}^n$. (In the figure above, T is such an object: it is the 1-dimensional boundary of a 2-dimensional subset of the plane.) In this "co-dimension 1 boundary" case, we can use a variety of algorithms to find the minimal decompositions. One efficient method is the graph-cut method which has been used to to calculate good approximations to the minimal decompositions for shapes in $\mathbb{R}^2$[6].

Example 7 (Differences between Shapes). *Now we are in a position to consider distances between shapes. In the figure, we notice that while a direct subtraction of T_1 from T_2 produces something whose mass is not small due to the fact that they do not coincide and cancel, the flat norm still sees the two currents as very close and assigns a small distance between T_1 and T_2.*
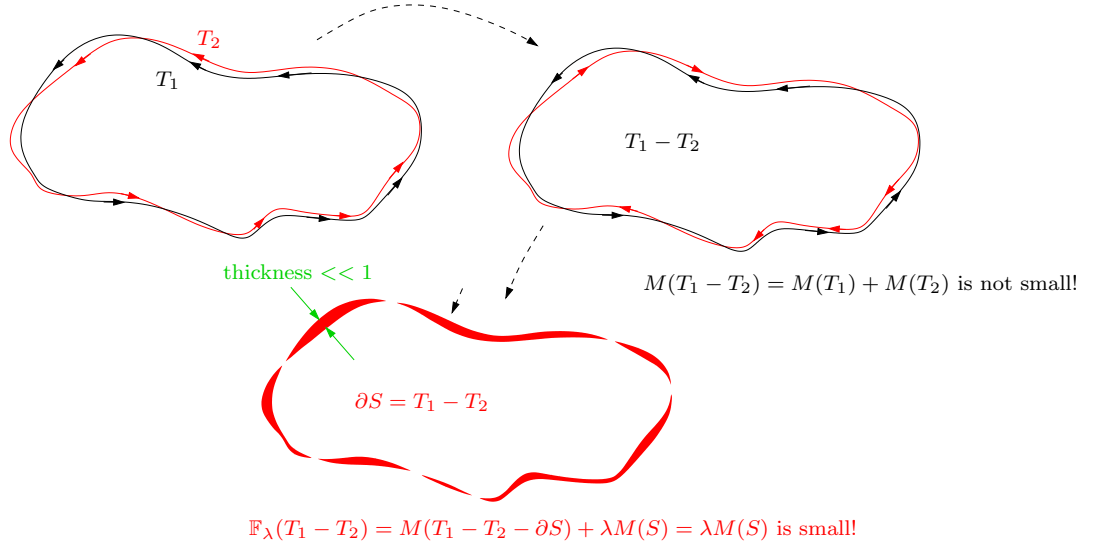
Now we are ready for the main result in the paper we are aiming to explain.

When does Integral T → Integral S?

Using currents to represent shapes, and the flat norm to measure size and distances in the space of currents leads us to the question:

Is the S minimizing $M(T - \partial S) + \lambda M(S)$ nice when the input T is nice?

In our case, we consider a current to be nice if it is an *integral current*. These correspond to unions of pieces of smooth[3] k-dimensional surfaces in $\mathbb{R}^n$, such that the k-dimensional volume of the union and the (k-1)-dimensional volume of its boundary are both finite. So the question can be restated:

---

[3]Actually, by smooth we mean that it is C^1, that is the surface is a submanifold of $\mathbb{R}^n$ whose tangents vary continuously.

To get to the answer, we introduce a bit more terminology: We say a set $E \subset \mathbb{R}^n$ has co-dimension k if $\dim(E) = n - k$.

Example 8 (co-dimension). *A 1-dimensional curve in $\mathbb{R}^2$ has co-dimension 1, whereas a 1-dimensional curve in $\mathbb{R}^3$ has co-dimension 2.*

Previous to this work what was known could be summed up in two statements:

1. Together [1, 5] show that when T is a co-dimension 1 boundary in $\mathbb{R}^n$, there is a minimizing integral S[4].

2. In [2] it is shown that co-dimension 1 T's have minimizing integral S's in the case that the T and S live on a *simplicial complex*. Note that T need not be a boundary, though it must be a discrete, simplicial current!

The Punchline

The main result of the paper by Ibrahim, Krishnamoorthy and Vixie[3] uses the results for discrete, simplicial currents to prove that the flat norm decompositions of arbitrary co-dimension 1 integral currents are also well behaved:

In essence, the argument for the main result boils down to the observation that if T is integral and there is not an integral minimizer then this implies, after lots of fussing around and using modified deformation theorems and special simplicial refinements and adaptations, that we can produce a integral simplicial current that also does not have an integral minimizer. And that is a contradiction, according to the results in [2].

The paper also uses the isoperimetric inequality to prove that when T is a boundary and λ is small enough, the flat norm minimization becomes a minimal surface problem. This is used to conclude that:

---

[4]We do not have uniqueness – there are times in which there are also non-integral minimizers along with the integral minimizer.

[5]We prove a case of the conjecture that gives us the result for 1-currents in $\mathbb{R}^2$. The full conjecture seems true, but may be difficult to prove.

because of examples of 1-dimensional boundaries in $\mathbb{R}^4$ for which there are spanning currents[6] S such that $M(S) < M(E)$ for any integral E with $\partial E = T$ [4].

The case of co-dimension 2 is completely open, except for the fact that we know we cannot use minimal surface problems to find counterexamples [4].

Post Script

While it is true that integral currents are *much* more regular than arbitrary currents, they are *not* almost smooth manifolds. In order to dispel that notion, we give some examples of nice but non-trivial integral 1-currents in $\mathbb{R}^2$. While it is true that there can only be a finite
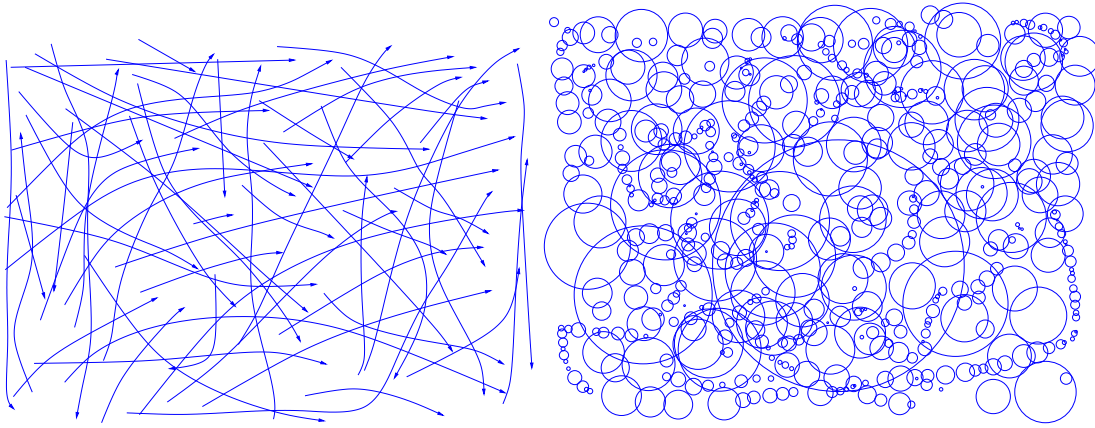
(c) A perfectly nice integral 1-current. Since the boundary has to be finite, there must be at most a finite number of non-closed curves since each such curve contributes 2 to the mass of the boundary.

(d) Another perfectly nice integral current. The circles are all oriented in the counterclockwise direction.

number of non-closed curves in an integral 1-current, that doesn't mean such a 1-current is simple. A current that is the union of 100,000 curves each with length of 2 or less will have a 1-dimensional volume of at most 200,000 and a 0-dimensional boundary volume of 200,000. Since both of these numbers are less than ∞, the current is an integral current.

An integral 1-current can have an infinite number of circles as long as the sum of the circumferences of the circles is finite. For example, the following is an integral current. Begin by enumerating all the points in the unit square having rational coordinates, $p_1 = (x_1, y_1), p_2 = (x_1, y_2), \dots$. Now let $C(x, r)$ be the circle centered at x with radius r, oriented counterclockwise. Finally, define

$$T = \bigcup_{i=1}^{\infty} C\left(p_i, \frac{1}{2^i}\right).$$

---

[6]S such that $\partial S = T$

7

We then have that $M(T) = 2\pi$ and $M(\partial T) = 0$, so T is a "nice" integral 1-current.

In the case of k-currents in $\mathbb{R}^n$, $n > 2$, things can be even more interesting since we can now have an infinite number of pieces having boundary ("non-closed" pieces) as long as the total boundary mass is finite.

Acknowledgment

References

[1] Tony F. Chan and Selim Esedoḡlu. Aspects of total variation regularized L^1 function approximation. *SIAM J. Appl. Math.*, 65(5):1817–1837, 2005.

[2] Sharif Ibrahim, Bala Krishnamoorthy, and Kevin R. Vixie. Simplicial flat norm with scale. *Journal of Computational Geometry*, 4(1):133–159, 2013. arxiv:1105.5104.

[3] Sharif Ibrahim, Bala Krishnamoorthy, and Kevin R. Vixie. Flat norm decomposition of integral currents. *arXiv*, 2014. http://arxiv.org/abs/1411.0882.

[4] Frank Morgan. Area-minimizing currents bounded by higher multiples of curves. *Rendiconti del Circolo Matematico di Palermo*, 33(1):37–46, 1984.

[5] Simon P. Morgan and Kevin R. Vixie. L^1TV computes the flat norm for boundaries. *Abstract and Applied Analysis*, 2007:Article ID 45153, 14 pages, 2007. doi:10.1155/2007/45153.

[6] Kevin R. Vixie, Keith Clawson, Thomas J. Asaki, Gary Sandine, Simon P. Morgan, and Brandon Price. Multiscale flat norm signatures for shapes and images. *Applied Mathematical Sciences*, 4(14):667–680, 2010.